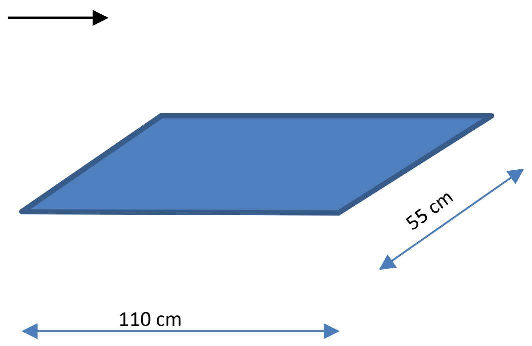
Laminar boundary Layer problem examples:

1) A thin flat plate 55cm by 110 cm is immersed in a 6m/s stream of oil with density 870kg/m^3 and viscosity 0.104 kg/ms. If the flow is parallel to the long side of the plate, what is Reynolds number at the end of the plate? Is the boundary layer at the end of the plate laminar or turbulent? Why?
   Ans: 5.52×10^4, laminar



Reynolds number is based on distance along the plate: $R_x = \frac{\rho U x}{\mu}$

At the end of the plate L is 110 cm = 1.1 m

$$R_L = \frac{\rho U L}{\mu} = \frac{870 \times 6 \times 1.1}{0.104} = 5.52 \times 10^4$$

An accepted value for transition for boundary layers is $R_{x,trans} = 5 \times 10^5$

As $5.52 \times 10^4 < 5 \times 10^5$ the flow can be assumed to be laminar.

1-1) Compute the total friction drag on one side of the plate if the stream is parallel to a) the long side and b) the short side.
Ans: a) 53.6 N and b) 75.7 N

a) L=1.1m, b= 0.55m

From the above result, we know the boundary layer is laminar.

Calculate C_D using correlation for laminar boundary layer (Blasius solution):

$$C_D = \frac{1.328}{R_L^{0.5}} = \frac{1.328}{(5.52 \times 10^4)^{0.5}} = 0.00565$$

Use C_D to find drag on one side of plate:

$$C_D = \frac{D/bL}{\frac{1}{2}\rho U^2}$$

$$D = 0.00565 \times (0.5 \times 870 \times 6^2) \times 0.55 \times 1.1 = 53.6N$$

b) L=0.55m, b= 1.1m

Calculate R_L:

$$R_L = \frac{\rho UL}{\mu} = \frac{870 \times 6 \times 0.55}{0.104} = 2.76 \times 10^4$$

As R_L is less than 5×10^5 we can assume the boundary layer is laminar

Calculate C_D using correlation for laminar boundary layer (Blasius solution):

$$C_D = \frac{1.328}{R_L^{0.5}} = \frac{1.328}{(2.76 \times 10^4)^{0.5}} = 0.008$$

Use C_D to find drag on one side of plate:

$$C_D = \frac{D/bL}{\frac{1}{2}\rho U^2}; \;\; D = 0.008 \times (0.5 \times 870 \times 6^2) \times 0.55 \times 1.1 = 75.7N$$

1*)

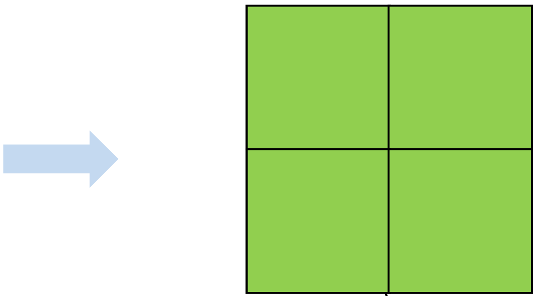
- In part 1-1)-a of the above problem consider that the value of the drag force have changed to 40N. Under this new flow condition find how much have changed the value of the velocity field above the plate.
  Answer: $U = 4.94\ m/s$

- If the value of the velocity field remains equal to $U = 6m/s$, how much the length of the plate has to be reduced in order that the drag force is 40N.
  Answer: $L = 0.614\ m = 61.4\ cm$

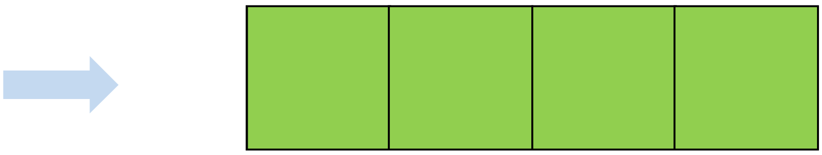
- Finally, if you are doing an experiment and want to have the original plate (L=1.1m, b= 0.55m) immersed in a 6m/s stream of oil and supporting the drag force of $40N$. How much you have to change the density of the oil, ρ, keeping the same value of the oil viscosity (μ =0.104 kg/ms) in order to reproduce the given value of the drag force.
  Answer: $\rho = 485.488 \frac{kg}{m^3}$, $R_L = 3.082 \times 10^4$ (laminar flow)

2) Consider laminar boundary layer flow past the thin, square plate arrangements below. Compared to the friction drag of a single plate, how much larger is the drag of 4 plates together as in configurations a) and b)?

Ans: $2.83F_1$, $2F_1$



a)



b)

length of side L=b=*d*, Free stream velocity, *U*

$$R_L = \frac{\rho U d}{\mu}$$

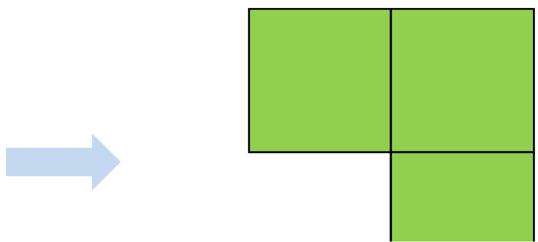
For laminar flow (Blasius solution), drag is found from:

$$C_D = \frac{D/bL}{\frac{1}{2}\rho U^2} = \frac{1.328}{R_L^{0.5}} \quad with \;\; A = bL = d^2 \;\; and \; D = \tfrac{1}{2}\rho U^2 A\, C_D \; (one\ side)$$

Let the drag force <u>from both sides</u> of the single plate be D_1:

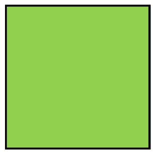
$$D_1 = 2\left(\tfrac{1}{2}\rho U^2\right)d^2\left(\frac{1.328}{\sqrt{\frac{\rho U d}{\mu}}}\right)$$

a) length of side L=b=2*d*, Free stream velocity, *U*



$$R_L = \frac{2\rho U d}{\mu}$$

For laminar flow, drag is found from:



$$C_D = \frac{D/bL}{\frac{1}{2}\rho U^2} = \frac{1.328}{Re_L^{0.5}}$$

So the drag force from both sides of this plate configuration is:

$$D_a = 2\left(\frac{1}{2}\rho U^2\right)4d^2\left(\frac{1.328}{\sqrt{\frac{2\rho U d}{\mu}}}\right) = \frac{4}{\sqrt{2}}D_1 = 2.83D_1$$

b) length of side *L=4d* and *b=d,* Free stream velocity, *U*

$$R_L = \frac{4\rho U d}{\mu}$$

For laminar flow, drag is found from:

$$C_D = \frac{D/bL}{\frac{1}{2}\rho U^2} = \frac{1.328}{Re_L^{0.5}}$$

So the drag force from both sides of this plate configuration is:

$$D_b = 2\left(\frac{1}{2}\rho U^2\right)4d^2\left(\frac{1.328}{\sqrt{\frac{4\rho U d}{\mu}}}\right) = \frac{4}{2}D_1 = 2D_1$$

2*) c) In the above problem the dimension of the original plate have changed to *L= d* and *b=d/2,* how much smaller is the drag is compared with the one corresponding to the original plate.
Answer: $D_c = D_1/2$

3) For flow of air moving at 4 m/s past a thin flat plate, by considering that the flow is laminar (Blasius solution) estimate the distances *x* from the leading edge at which the boundary layer thickness will be 1 mm. For the obtained value of the distance *x* find the corresponding Reynolds number, surface shear, momentum thickness and drag force by unit width.

For air, take ρ = 1.2 kg/m³ and μ= 1.8× 10^{-5} kg/(m s).

Problem parameters:

$$\rho = 1.2\frac{kg}{m^3}; U = 4\frac{m}{s};\ \delta = 1mm = 10^{-3}m;\ \mu = 1.8\ \times 10^{-5}\frac{kg}{m\,s}$$

Blasius solution:

Boundary layer thickness

$$\frac{\delta}{x} = \frac{5}{\sqrt{R_x}}; \;\; R_x = \frac{\rho U x}{\mu}$$

From above expressions follows that

$$x = \frac{\delta^2 \rho\, U}{25\mu} = \frac{10^{-6} \times\; 1.2 \times\; 4}{25\; \times 1.8 \times 10^{-5}} = 0.1066\; \times 10^{-1} m$$

Then

$$R_x = \frac{\rho U x}{\mu} = \frac{1.2\; \times 4 \times 0.1066 \times 10^{-1}}{1.8 \times 10^{-5}} = 0.284 \times 10^{4};\;\; {R_x}^{1/2} = 0.533 \times 10^{2}$$

Surface shear stress

$$\tau_0 = \rho U^2 \frac{0.332}{\sqrt{R_x}} = 11.96 \times 10^{-2}\; \frac{kg}{m\, s^2}$$

Momentum thickness $at\; x = 0.1066\; \times 10^{-1} m$

$$\theta = \frac{0.664\, x}{\sqrt{R_x}} = 0.133\; \times 10^{-3} m;$$

From momentum valance along the plate the drag per unit width until $x = 0.1066\; \times 10^{-1} m$

$$\frac{D}{b} = \rho U^2 \theta = 2.56 \times 10^{-3} \frac{kg}{s^2}$$

3*) If in the above problem instead of obtaining the value of x at which the boundary layer thickness is δ =1 mm, you want to know the value of boundary layer thickness at a distance distances *x=0.02m* from the leading edge of the plate. Find at that position from the leading edge of the plate find the corresponding value of the surface shear, displacement and momentum thicknesses.

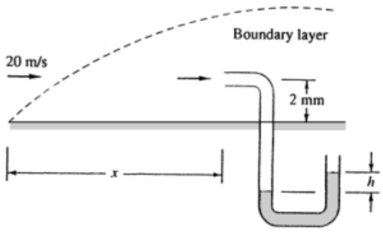
Answer: $\delta = 0.1369 \times 10^{-2} m,\;\; \tau_0 = 8.73 \frac{kg}{ms^2}, \theta = 0.182 \times 10^{-3} m$

4) Air at 20°C and 1 at flows at 10 m/s past the flat plate in the figure. A pitot stagnation tube, placed 2 mm from the wall, develops a manometer head *h* = 4 mm of Meriam red oil, SG = 0.827. Use this information to estimate the downstream position x of the pitot tube and boundary layer thickens at that position. For the obtained value of the distance *x* find the corresponding Reynolds number, surface shear, momentum thickness and drag force by unit width.

To find your solution, assume a von Karman laminar velocity profile.

For air, take ρ = 1.2 kg/m³ and μ= 1.8 10^{-5} kg/(m s).
Density of oil, ρ_o = 0.827× 998= 825.346 kg/m³

According to Bernoulli equation, the boundary layer velocity measured with the pitot tube at 2 mm from the wall is given by $\frac{1}{2}u^2(x, y = 2mm) = \frac{\rho_O g\, h}{\rho}$, where it has been considered that the flow over the plate is air.



Problem parameters:

$$\rho = 1.2\frac{kg}{m^3}; U = 10\frac{m}{s};\ y = 2mm = 2\ 10^{-3}m; h = 4mm = 4\ 10^{-3}m;\ \mu = 1.8\ 10^{-5}\frac{kg}{m\ s}$$

$\rho_o = 0.827\times 998 = 825.346\ \frac{kg}{m^3}$

Pitot tube

$$u(x, y = 2mm) = \left(\frac{2\ \rho_O g\ h}{\rho}\right)^{\frac{1}{2}} = 7.34\ \frac{m}{s} < 10$$

$$\frac{u(x, y = 2mm)}{U} = 0.734$$

von Karman velocity profile

$$\frac{u}{U} = 2\eta - \eta^2,\ with\ \eta = \frac{y}{\delta}$$

$$2\eta - \eta^2 = 0.734;\quad \eta = \frac{2mm}{\delta}$$

Quadratic equation

$$\eta = \begin{cases} 1.516 \\ 0.484 \end{cases}$$

$\eta = 1.516;\ y > \delta$ inconsistent

$$\delta = \frac{2mm}{\eta} = \frac{2mm}{0.484} = 4.14\times\ 10^{-3}m$$

von Karman boundary layer thickness

$$\frac{\delta}{x} = \frac{5.48}{\sqrt{R_x}};\ \text{ where }\ R_x = \frac{\rho U x}{\mu}$$

Then

$$x = \frac{\delta^2 \rho\ U}{30\mu} = 0.380\ m;\quad R_x = \frac{\rho U x}{\mu} = 25\ \times 10^4;\ {R_x}^{1/2} = 5\times\ 10^2$$

Surface shear

$$\tau_0 = \frac{1}{2}\rho U^2 \frac{0.73}{{R_x}^{1/2}} = 8.69\ \times 10^{-2}\ Pa$$

"Alternative"

$$\frac{u}{U} = 2\eta - \eta^2, \qquad with\ \ \eta = \frac{y}{\delta}$$

$$\frac{\partial u}{\partial y} = \frac{U}{\delta}(2 - 2\eta);$$

$$\tau_0 = \mu \frac{\partial u}{\partial y}\bigg|_{y=0} = \frac{\mu}{\delta}\frac{\partial u}{\partial \eta}\bigg|_{\eta=0} = 2\mu\frac{U}{\delta} = 8.69 \times\ 10^{-2}\ Pa$$

Momentum thickness

$$\theta = \frac{0.730\ x}{\sqrt{R_x}} = 0.555 \times\ 10^{-3} m$$

Drag force per unit width

$$\frac{D}{b} = \rho U^2 \theta = 0.666\ \times 10^{-1} N/m$$

4*) In the above problem, using the results obtained above, find at what distance from the wall the pitot stagnation tube has to be placed in order that velocity measured inside the boundary layer is $u = 5\ m/s$.

Answer: $y = 1.213 \times 10^{-3} m$

5) Using the four order polynomial velocity profile suggested by K. Pohlhausen in 1921, find the corresponding integrated boundary layer equations.

$$\frac{u}{U} = 2\eta - 2\eta^3 + \eta^4,\ with\ \ \eta = \frac{y}{\delta}$$

Note: This problem is for general knowledge and not to be considered in this module.

You can verify that the above velocity profile satisfies the following boundary conditions, and consequently is an improvement to von Karman solution:

At $y = \delta, i.e.\ \eta = 1$

$$u(x,\delta) = 1, \qquad \frac{\partial u(x,\delta)}{\partial y} = 0, \qquad and \ \frac{\partial^2 u(x,\delta)}{\partial y^2} = 0$$

At $y = 0 i.e.\ \eta = 0$

$$u(x,0) = 0, \quad and \ \frac{\partial^2 u(x,0)}{\partial y^2} = 0$$

In this case it is easy to verify that the Momentum thickness is given by (not necessary to proof):

$$\theta = \int_0^{\delta(x)} \frac{u}{U}\left(1 - \frac{u}{U}\right) dy = \frac{37}{315}\delta(x)$$

Knowing the relation between the momentum thickness θ and boundary layer thickness δ, and using the integral relation for a flat plate boundary layer flow found the corresponding expressions for $\delta(x), \tau_0(x), D(x)$ and $C_D(x)$.

To find an expression for $\delta(x)$, use the following definitions of the surface shear $\tau_0(x)$:

$$\tau_0(x) = \mu \frac{\partial u}{\partial y}\bigg|_{y=0} = \mu \frac{1}{\delta}\frac{\partial u}{\partial \eta}\bigg|_{\eta=0}$$

and according to the Boundary Layer depth average integrated momentum equation:

$$\tau_0 = \rho\, U^2 \frac{d\theta}{dx}$$

From the above two equations and the obtained value of θ, you can find a first order ordinary differential equation, which integral give you the expression of $\delta(x)$.

Knowing and the expression for $\delta(x)$ you can find the values of

$$\frac{D}{b} = \rho U^2 \theta, \quad C_D = \frac{(D/bx)}{(\rho U^2/2)} = 2\frac{\theta}{x} \ \ and\ \tau_0(x)\ as\ function\ x\ and\ \ R_x = \frac{\rho U x}{\mu}$$

K. Pohlhausen velocity profile

$$\frac{u}{U} = 2\eta - 2\eta^3 + \eta^4, \qquad with\ \ \eta = \frac{y}{\delta}$$

$$\frac{\partial u}{\partial y} = \frac{U}{\delta}(2 - 6\eta^2 + 4\eta^3)$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{U}{\delta^2}(-12\eta + 12\eta^2)$$

You can verify that the above profile satisfies the boundary conditions:

At $y = \delta, i.e.\ \eta = 1$

$$u(x,\delta) = 1, \qquad \frac{\partial u(x,\delta)}{\partial y} = 0, \qquad and\ \frac{\partial^2 u(x,\delta)}{\partial y^2} = 0$$

At $y = 0 i.e.\ \eta = 0$

$$u(x,0) = 0, \quad and\ \frac{\partial^2 u(x,0)}{\partial y^2} = 0$$

Momentum thickness

$$\theta = \int_0^{\delta(x)} \frac{u}{U}\left(1 - \frac{u}{U}\right) dy = \frac{37}{315}\delta(x)$$

Surface shear

$$\frac{\partial u}{\partial y} = \frac{U}{\delta}(2 - 6\eta^2 + 4\eta^3)$$

$$\tau_0 = \mu \frac{\partial u}{\partial y}\bigg|_{y=0} = \frac{\mu}{\delta}\frac{\partial u}{\partial \eta}\bigg|_{\eta=0} = 2\mu\frac{U}{\delta}$$

and

$$\tau_0 = \rho\, U^2 \frac{d\theta}{dx} = \rho\, U^2 \frac{37}{315}\frac{d\delta}{dx}$$

Therefore

$$2\mu\frac{U}{\delta} = \rho\, U^2 \frac{37}{315}\frac{d\delta}{dx}; \quad \frac{1}{2}\frac{d\delta^2}{dx} = \frac{630\mu}{37\rho U}; \ \delta^2 = \frac{1260\mu x}{37\rho U}$$

Boundary layer thickness

$$\delta = \left(\frac{34.05\mu x}{\rho U}\right)^{1/2} = 5.83\frac{x}{{R_x}^{1/2}}$$

Drag force per unit width

$$\frac{D}{b} = \rho U^2 \theta = \rho U^2 \frac{37}{315}\delta = 0.684\frac{\rho U^2 x}{{R_x}^{1/2}}$$

$$C_D = 2\frac{\theta}{x} = \frac{1.372}{{R_x}^{1/2}}$$

and

$$\tau_0 = \rho\, U^2 \frac{d\theta}{dx} = \rho\, U^2 \frac{37}{315}\frac{d\delta}{dx}; \quad \delta = \left(\frac{34.05\mu x}{\rho U^2}\right)^{1/2}$$

$$\tau_0 = 0.685\rho^{1/2}U\mu^{1/2}\frac{dx^{1/2}}{dx} = 0.342\frac{\rho^{1/2}U\mu^{1/2}}{x^{1/2}} = 0.342\frac{\rho U^2}{{R_x}^{1/2}}$$

Turbulent boundary Layer problem examples:

6) A thin flat plate 55 by 110 cm is immersed in a 6-m/s stream of water at 20°C. Compute the total friction drag if the stream is parallel to (a) the long side, L=110 cm and b=55 cm, and (b) the short side, L=55 cm and b=110 cm.

Consider that the flow is turbulent and use Prandtl turbulent boundary layer approximation.

For water at 20°C, take ρ = 998 kg/m^3 and μ = 0.001 kg/m s.

Problem parameters:

$$\rho = 998\frac{kg}{m^3};\ \ \mu = 10^{-3}\frac{kg}{m\ s};\ U = 6\frac{m}{s}; L = 1.10m; b = 0.55m$$

(a) The long side, L=1.10 m, and $b = 0.55m$

$$R_L = \frac{\rho UL}{\mu} = 6.59\ \times 10^6$$

Prandtl turbulent boundary layer approximation:

$$R_L^{\frac{1}{7}} = 9.42$$

Drag

$$C_D = \frac{0.031}{(R_L)^{1/7}} = 3.29\times\ 10^{-3}$$

Area=b L

$$D^* = \frac{1}{2}\rho U^2(b\ L)C_D\ \ = 35.755\ kg\frac{m}{s^2}\ (N);\ \ (1\text{ side});$$

$$D = 2\ D^* = 71.51\ N;\ \ (2\text{ sides})$$

(b) The long side, L=0.55 m, and $b = 1.10\ m$

$$R_l = \frac{\rho UL}{\mu} = 3.29\times\ 10^6;\ \ R_L^{\frac{1}{7}} = 8.53$$

$$C_D = \frac{0.031}{(R_L)^{1/7}} = 3.63\ \times 10^{-3}$$

$$D^* = \frac{1}{2}\rho U^2 b\, l\, C_D = 39.45\, kg \frac{m}{s^2}\ (N); \quad (1 \text{ side});$$

$$D = 2\, D^* = 78.90\, N; \ \ (2 \text{ sides})$$

6*) In problem 6-a, find the thickness of the boundary layer δ at the tailing edge of the plate, the Reynolds number in terms of δ, i.e. R_δ, the sink friction coefficient C_f, and the laminar sublayer thickness δ' at the tailing edge of the plate. If the surface roughness $\varepsilon' = 5 \times 10^{-6} m$, how do you classify the surface of the plate as hydraulic smooth or rough?

Answer: $\delta = 0.0189m$; $R_\delta = 0.1132 \times 10^{-6}$; $C_f = 0.2816$; $\delta' = 1.78 \times 10^{-6} m$. Thefore the surface is hydraulic rough since $\varepsilon' > \delta'$

7) A ship is 125 m long and has a wetted area of 3500 m². Its propellers can deliver a maximum power of 1.1 MW to seawater at 20°C. If all drag is due to friction, estimate the maximum ship speed.

Power=Force × velocity= D ×U=1.1 $\times 10^6\ Watts$

For water at 20°C, take ρ = 1025 kg/m³ and μ = 0.00107 kg/m s.

Consider that the flow is turbulent and use Blasius turbulent boundary layer approximation.

Problem parameters:

$$\rho = 1025 \frac{kg}{m^3}; \ \ \mu = 1.07\ \times 10^{-3} \frac{kg}{m\, s}; \ L = 125\, m; A = 3500\, m^2$$

$L = 125\, m$ and $A = 3500\, m^2$

Ship speed ***U*** ??? $R_L = \frac{\rho U L}{\mu} = 1.197 \times\ 10^8\ U$

Blasius turbulent boundary layer approximation:

$$C_D = \frac{0.072}{(R_L)^{1/5}} = 1.745\ 10^{-3}\ U^{-1/5}$$

$$D = \frac{1}{2}\rho U^2 (b\, L)\, C_D = \frac{1}{2}\rho U^2 A\, C_D = 3130.09\, U^2\, U^{-1/5}$$

Power = Force × velocity = $D \times\ U$ =1.1 × $10^6\ Watts\ (kg\, m^2 s^{-3})$

$$(3130.09\, U^2\, U^{-1/5})\, U = 1.1\ \times 10^6; \ \ U^{14/5} = 351.42$$

$$U = 8.11\ \frac{m}{s}$$

7*) If in the above problem, the force (drag) required to move the ship is $D = 100 \times 10^3 \frac{kg\ m}{s^2} = 100 \times 10^3 N$, estimate how much is the ship speed and the power deliberate by the propellers.

Answer: $U = 6.85 \frac{m}{s}; P = 0.685 \times 10^6\ \frac{kg\ m^2}{s^3} = 0.685\ MW$

8) An alternate way of obtaining Prandtl (1927) turbulent flat-plate flow integrated equations is by using the following wall shear-stress formula.

$$\tau_0 = 0.0225\ \rho\ U^2 \left(\frac{\mu}{\rho U \delta}\right)^{1/4}$$

Use this information to find the corresponding expressions for the boundary layer thickens, surface shear, momentum thickness and drag force in terms of the Reynolds number.

Note: This problem is for general knowledge and not to be considered in this module.

In your solution as in the original Prandtl analysis use the 1/7th power law turbulent velocity profile approximated, i.e.

$$\frac{u}{U} = \left(\frac{y}{\delta}\right)^{1/7}$$

Therefore

$$\theta = \int_0^{\delta(x)} \frac{u}{U}\left(1 - \frac{u}{U}\right) dy = \frac{7}{72}\delta$$

Given surface shear

$$\tau_0 = 0.0225\ \rho\ U^2 \left(\frac{\mu}{\rho U \delta}\right)^{1/4}$$

Also from Momentum integral

$$\tau_0 = \rho\ U^2 \frac{d\theta}{dx}$$

Therefore

$$0.0225\ \rho\ U^2 \left(\frac{\mu}{\rho U \delta}\right)^{1/4} = \rho\ U^2 \frac{d\theta}{dx} = \rho\ U^2 \frac{7}{72}\frac{d\delta}{dx}$$

or

$$0.231\left(\frac{\mu}{\rho U\delta}\right)^{1/4} = \frac{d\delta}{dx};\ \text{ then }\ 0.231\left(\frac{\mu}{\rho U}\right)^{1/4} = \delta^{1/4}\frac{d\delta}{dx} = \frac{4}{5}\frac{d\delta^{5/4}}{dx}$$

Integrating

$$0.289\left(\frac{\mu}{\rho U}\right)^{1/4} = \frac{d\delta^{5/4}}{dx}\ \rightarrow \delta^{5/4} = 0.289\left(\frac{\mu}{\rho U}\right)^{1/4} x$$

Therefore

$$\delta = 0.37\left(\frac{\mu}{\rho U}\right)^{1/5} x^{4/5} = \frac{0.37}{(R_x)^{1/5}};\quad R_x = \frac{\rho U x}{\mu}$$

Having obtaining δ, follows:

$$\frac{\theta}{x} = \frac{7}{72}\delta = \frac{0.036}{(R_x)^{1/5}}$$

$$\tau_0 = 0.0225\,\rho\,U^2\left(\frac{\mu}{\rho U\delta}\right)^{\frac{1}{4}} = \frac{0.058\frac{1}{2}\rho U^2}{(R_x)^{\frac{1}{5}}}$$

$$D = \rho b U^2\theta = \rho(b\,x)U^2\frac{0.036}{(R_x)^{1/5}}$$

Attending a tutorial meeting